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Crux of the Clock Paradox

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Abstract

The central riddle or crux of the relativistic clock paradox is studied with methods that are familiar from thermodynamics and hamiltonian mechanics, but which are not usually used, as here, to relate theory to phenomenology in the context of relativistic chronometry. Contrasting ramifications of the special and general theory are considered.

1. *Introduction*

The relativistic clock paradox can be analyzed in various different geometries (Markley, 1973; Brans & Stewart, 1973), but the central riddle or crux of the problem is most readily demonstrated for the case of one-dimensional motion with a two-dimensional space-time.

This case has been exhaustively analyzed in special relativity (Rindler, 1969a; Schild, 1959), but there are significant results which can be derived without invoking the geometric structure of space-time.

This is done here by using mathematical methods which have proved very useful in thermodynamics and hamiltonian mechanics, but which are not commonly used in the context of relativistic chronometry (Synge, 1960a).

For equations (2.1)–(2.15), special relativity is implicitly assumed, but the Lorentz metric is not introduced explicitly except where necessary, thus indicating more specifically the role of the metric, as opposed to results which follow from the properties of coordinate transformations.

Equations (3.1)–(3.2) are general relativistic to lowest order terms in a gravitational potential V which uniquely cancels the time dilatation, to give a striking example of how the clock paradox can vanish in the absence of non-gravitational forces, however real it may be under other conditions.

2. *Special Relativity*

This is essentially the problem of two clocks, in motion relative to one another, but fixed in their respective rest frames, one of which has the time coordinate t and spatial coordinate x, while the other has a time coordinate T and spatial coordinate X.

In special relativity the pseudo-Euclidean space-time (Minkowski space) is assumed to be globally mapped out by cartesian coordinates, and it is assumed that there is a Lorentz transformation relating the coordinate pair (t, x) to the pair (T, X) (although this need not necessarily be true in general) (Wheeler, 1962).

These assumptions are far-reaching, and have non-trivial consequences, one of which can be identified with the clock paradox.

Once it is assumed that there is a functional relationship between (t, x) and (T, X), and that this relationship is a differentiable mapping, the coordinate transformation can conveniently be expressed in a differential form, as a relationship among the coordinate differentials (dt, dx) and (dT, dX), as,

$$dT = (\partial T/\partial t)_x\, dt + (\partial T/\partial x)_t\, dx \tag{2.1a}$$
$$= (\partial T/\partial t)_X\, dt + (\partial T/\partial X)_t\, dX \tag{2.1b}$$
$$= (\partial T/\partial x)_X\, dx + (\partial T/\partial X)_x\, dX \tag{2.1c}$$
$$dX = (\partial X/\partial t)_x\, dt + (\partial X/\partial x)_t\, dx \tag{2.2a}$$
$$= (\partial X/\partial t)_T\, dt + (\partial X/\partial T)_t\, dT \tag{2.2b}$$
$$= (\partial X/\partial T)_x\, dT + (\partial X/\partial x)_T\, dx \tag{2.2c}$$
$$dt = (\partial t/\partial T)_X\, dT + (\partial t/\partial X)_T\, dX \tag{2.3a}$$
$$= (\partial t/\partial T)_x\, dT + (\partial t/\partial x)_T\, dx \tag{2.3b}$$
$$= (\partial t/\partial X)_x\, dX + (\partial t/\partial x)_X\, dx \tag{2.3c}$$
$$dx = (\partial x/\partial T)_X\, dT + (\partial x/\partial X)_T\, dX \tag{2.4a}$$
$$= (\partial x/\partial T)_t\, dT + (\partial x/\partial t)_T\, dt \tag{2.4b}$$
$$= (\partial x/\partial t)_X\, dt + (\partial x/\partial X)_t\, dX \tag{2.4c}$$

where the notation is borrowed from thermodynamics, so that, in equation (2.1a), e.g., T is treated as a function of t and x, i.e., $T = T(t, x)$, and $(\partial T/\partial t)_x$ is the partial derivative of T with respect to t, when x is held constant; and the other coefficients are similarly defined in terms of their independent variables, whose differentials are indicated explicitly on the right-hand side of each equation, indicating that each of the above twelve equations uses a different combination of dependent and independent variables.

As in thermodynamics, the physical content is determined by the phenomenological significance of the coefficients.

If one clock is fixed at constant x, and is moving with velocity w relative to the other clock, which is assumed to be fixed at constant X (in the frame of reference using time coordinate T), then equation (2.2c) gives,

$$w = (\partial X/\partial T)_x \tag{2.5}$$

a relation valid for cartesian coordinates (though not necessarily for generalized coordinates).

Similarly, the clock at constant X has a velocity W relative to the clock at constant x, and equation (2.4c) gives,

$$W = (\partial x/\partial t)_X \tag{2.6}$$

It is also readily reasoned out that the Lorentz–Fitzgerald length contraction factor (Rindler, 1969b), for a rod of infinitesimal length located at fixed x, and moving with the velocity w relative to the clock at fixed X, is given by equation (2.2c) as,

$$f = (\partial X/\partial x)_T \tag{2.7}$$

For an infinitesimal rod at fixed X, with velocity W relative to the clock at fixed x, equation (2.4c) gives the contraction factor,

$$F = (\partial x/\partial X)_t \tag{2.8}$$

If chronometric asymmetry is assumed, i.e., if it is assumed that the clocks go at different rates due to relativistic effects (but not for any other reason), then it can be assumed, for the sake of definiteness, that the clock at fixed X goes faster than the clock at fixed x. In special relativity, the proper time of a clock (Synge, 1960a), i.e., the time actually registered by the clock, is the same as the coordinate time of its rest frame (the co-moving frame of the clock); this is a consequence of using the Lorentz metric with cartesian coordinates (again it is not necessarily true in generalized coordinates). Equation (2.1a) then gives the time dilatation factor as,

$$\Gamma = (\partial T/\partial t)_x \tag{2.9}$$

when the problem is analyzed relative to the co-moving frame of an observer at fixed x (at the position of the slower clock). On the other hand, when the problem is analyzed relative to the co-moving frame of an observer at fixed X (at the position of the faster clock), equation (2.3a) gives the time 'slowdown' factor,

$$\gamma = (\partial t/\partial T)_X \tag{2.10}$$

for the relative rate of the slower clock, which has the velocity w relative to the frame of reference with coordinates (T, X).

If Γ and γ are what they are supposed to be, i.e., the time dilatation and slowdown factors, respectively, giving the relative rates of the two clocks, then common sense would appear to suggest that they should be mutually reciprocal. The crux of the clock paradox is that there is no such reciprocity, i.e.,

$$\Gamma\gamma \neq 1 \tag{2.11}$$

contrary to intuition. As Rindler (1969) emphasizes, Γ and γ are actually equal, rather than reciprocal, so that the reference to γ as a 'slowdown' factor is a misnomer (it would be a misconception to treat γ that way).

These results can be derived explicitly (without making explicit use of the metric) by noting that, although Γ and γ are not mutually reciprocal, yet equations (2.3)–(2.4) represent the reciprocal transformation of equations (2.1)–(2.2). This then yields the identity,

$$(\partial X/\partial t)_x(\partial x/\partial X)_t + (\partial x/\partial t)_X = 0 \qquad (2.12)$$

which, according to equations (2.5)-(2.10), is equivalent to the result,

$$\Gamma F = -W/w \qquad (2.13)$$

and it is similarly deduced that,

$$\gamma f = -w/W \qquad (2.14)$$

so that equations (2.13) and (2.14) give,

$$\Gamma\gamma = (Ff)^{-1} \qquad (2.15)$$

So far, the metric has only been invoked for the interpretation of Γ and γ in equations (2.9) and (2.10). It now becomes necessary to invoke it again, by noting that the Lorentz metric (the pseudo-Euclidean metric of Minkowski space) guarantees that the length contraction only works one way, i.e., a moving rod never manifests a length dilation; hence $Ff < 1$, and equation (2.15) $\Rightarrow \Gamma\gamma > 1$, thus refining equation (2.11), in a manner which indicates that the result derived here for infinitesimal differentials will also hold for arbitrary finite time intervals (such as those involved in the round-trip problems for which the clock paradox is typically considered), because an arbitrary sum or integral of positive infinitesimals is necessarily positive, and therefore there is no chance that the net effect will cancel out macroscopically, although equation (2.11) in itself does not suffice to prove this.

The Lorentz metric also gives $W = -w$, so that equations (2.13) and (2.14) give the result expected from the invariance of the two-dimensional space-time volume element $dt\,dx$ under Lorentz transformations.

3. *General Relativity*

In general relativity the fundamental difference between gravitation and non-gravitational forces must be reckoned with (Synge, 1960b), and is exemplified in the first-order formula of Hafele & Keating (1972) for the fractional change (δ) in the proper time rate of a clock moving with speed v at gravitational potential V,

$$\delta = V - \tfrac{1}{2}v^2 \qquad (3.1)$$

to lowest-order terms, using units in which the vacuum speed of light $c = 1$, and noting that, to this order, for a clock of unit mass, $\frac{1}{2}v^2$ is the kinetic energy K.

If (e.g.) V is a harmonic oscillator potential (as will be the case for minimal oscillations of a clock suspended at the end of a pendulum), then K, V, and δ can be averaged over a cycle, and the averaged quantities $\bar{K}$, $\bar{V}$, and $\bar{\delta}$ satisfy,

$$\bar{\delta} = \bar{V} - \bar{K} \qquad (3.2)$$

For a harmonic oscillator, it is well known that $\bar{V} = \bar{K}$, and hence $\bar{\delta} = 0$, i.e., the proper time rate of such an oscillating clock is independent of the amplitude of oscillation, except for transient variations which cancel out on the average, thus giving at least one example of a result which is consistent with the theory of Sachs (1973).

The possibility of such cancellation, together with examples of non-cancellation (Markley, 1973; Brans & Stewart, 1973), complicates the general relativistic time dilatation in a manner which has not yet been completely and unambiguously resolved.

References

Hafele, J. C. and Keating, R. E. (1972). *Science, New York,* **177**, 166. Hafele, J. C. (1972), *American Journal of Physics,* **40**, 81.

Markley, F. L. (1973). *American Journal of Physics,* **41**, 1246. Brans, C. H. and Stewart, D. R. (1973). *Physical Review,* **D8**, 1662.

Rindler, W. (1969a). *Essential Relativity,* Section 35. Van Nostrand Reinhold, New York.

Rindler, W. (1969b). *Essential Relativity,* Sect. 32. Van Nostrand Reinhold, New York.

Sachs, M. (1973). *American Journal of Physics,* **41**, 748. (1971). *Physics Today* **24**, 23.

Synge, J. L. (1960a). *Relativity: The General Theory,* Chap. III, Sect. 2. North-Holland, Amsterdam.

Synge, J. L. (1960b). *Relativity: The General Theory,* Chap. III, Sect. 3. North-Holland, Amsterdam.

Wheeler, J. A. (1962). Allowed coordinates and transformations depend on the topology of the pseudo-Riemannian space-time manifold. *Geometrodynamics,* p. 35. Academic Press, New York.